# Scaling quasi-isometries and subgroups of wreath products

Vincent Dumoncel

IMJ-PRG, University of Paris

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# Quasi-isometries and wreath products

# Quasi-isometries

For a finitely generated group $G = \langle S_G \rangle$, we define the *word length* of an element $g \in G$ as

$$|g|_{S_G} := \min\{n \geq 0 : \exists s_1, \ldots, s_n \in S_G \cup S_G^{-1}, g = s_1 \ldots s_n\}$$

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**Definition**

Let $G$ and $H$ be two finitely generated groups. A map $f : G \longrightarrow H$ is a quasi-isometry if there exist $C \geq 1$, $K \geq 0$ such that

- $\frac{1}{C} \cdot d_G(g,h) - K \leq d_H(f(g), f(h)) \leq C \cdot d_G(g,h) + K$ for all $g, h \in G$ ;
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Equivalently, $f$ satisfies the first point above and there exists a map $g : H \longrightarrow G$ such that $d(g \circ f, \mathrm{Id}_G), d(f \circ g, \mathrm{Id}_H) \leq K$, where for two maps $h_1, h_2 : G \longrightarrow H$ defined over $G$, their distance is

$$d(h_1, h_2) := \sup_{g \in G} d_H(h_1(g), h_2(g)).$$

# Wreath products

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If in addition $G = \langle S_G \rangle$ and $H = \langle S_H \rangle$ are finitely generated, then $G \wr H$ is finitely generated as well, and the finite set

$$\{\delta_a : a \in S_G\} \cup S_H$$

generates $G \wr H$, where $\delta_a : H \longrightarrow G$ sends $1_H$ to $a \in G$ and any other $h \neq 1_H \in H$ to $1_G$.

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Question: When are $G_1 \wr H_1$ and $G_2 \wr H_2$ quasi-isometric?

# Genevois-Tessera's results

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**Theorem [GT21]**

Let $F_1$ and $F_2$ be non-trivial finite groups. Let $G_1$ and $G_2$ be finitely presented one-ended groups. Then the following hold.

- If $G_1$ is not amenable, then $F_1 \wr G_1$ and $F_2 \wr G_2$ are quasi-isometric if and only if $|F_1|$ and $|F_2|$ have the same prime divisors and there exists a quasi-isometry $G_1 \longrightarrow G_2$.
- If $G_1$ is amenable, then $F_1 \wr G_1$ and $F_2 \wr G_2$ are quasi-isometric if and only if there exist $a, r, s \geq 1$ such that $|F_1| = a^r$, $|F_2| = a^s$ and a quasi-$\frac{s}{r}$-to-one quasi-isometry $G_1 \longrightarrow G_2$.

# Scaling quasi-isometries

# Scaling QI

## Definition

Let $G, H$ be finitely generated groups and let $f : G \longrightarrow H$ be a quasi-isometry. Let $k > 0$. We say that $f$ is quasi-$k$-to-one if there exists $C > 0$ such that

$$\Big| k|A| - |f^{-1}(A)| \Big| \leq C \cdot |\partial_H A|$$

for all finite subsets $A \subset H$, where $\partial_H A := \{h \in H \setminus A : \exists a \in A, d_H(h, a) = 1\}$.

In that case, we say that $f$ is *measure-scaling*, and that $k$ is the *scaling factor*.

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One motivation for the notion is a famous result of Whyte, stating that:

**Theorem [Why99]**

Let $G$ and $H$ be finitely generated groups. A quasi-isometry $f : G \longrightarrow H$ is quasi-one-to-one if and only if it lies at bounded distance from a bijection.

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On the other hand, in amenable situations, the scaling factor is unique when it exists:

**Lemma [GT22]**

Let $f : G \longrightarrow H$ be a quasi-isometry between amenable finitely generated groups. If $f$ is quasi-$k$-to-one and quasi-$k'$-to-one, then $k = k'$.

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Main example to keep in mind: if $H \leqslant G$ has finite index, the inclusion $H \hookrightarrow G$ is quasi-$\frac{1}{[G:H]}$-to-one.

# Scaling QI

Additionally, we have the following stability properties:

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Let $G, H, K$ be three finitely generated groups. Let $f, h\colon G \longrightarrow H$, $g\colon H \longrightarrow K$ be quasi-isometries.

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- If $f$ is quasi-$k$-to-one, then any of its quasi-inverses is quasi-$\frac{1}{k}$-to-one.

Thus, when $G$ is amenable, there is a well-defined group morphism

$$\mathrm{Sc}\colon \mathrm{QI}_{\mathrm{sc}}(G) \longrightarrow (\mathbb{R}_{>0}, \cdot)$$

where $\mathrm{QI}_{\mathrm{sc}}(G) = \{\text{scaling QI } G \longrightarrow G\}/\text{bounded distance}$, sending $f \in \mathrm{QI}_{\mathrm{sc}}(G)$ to its scaling factor. The image of this morphism, denoted $\mathrm{Sc}(G)$, is called the *scaling group of $G$*.

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**Proposition [GT22]**

- $\mathrm{Sc}(\mathbb{Z}^d) = \mathrm{Sc}(\mathbb{R}^d) = \mathbb{R}_{>0}$ for all $d \geq 1$.
- $\mathrm{Sc}(\mathrm{BS}(1,n)) = \mathbb{R}_{>0}$ for any $n \geq 1$;
- $\mathrm{Sc}(\mathrm{SOL}(\mathbb{Z})) = \mathbb{R}_{>0}$.

# Scaling groups of lamplighters

On the other hand, for lamplighters over one-ended groups, we have:

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In particular, $\mathrm{Sc}(F \wr G) = \{1\}$ when $F$ is finite and $G$ is amenable, finitely presented and one-ended.

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**Theorem [Dum25+]**

Let $N$ and $M$ be polynomial growth groups, of growth degrees $n$ and $m$. Let $G$ and $H$ be finitely presented amenable groups from $\mathcal{M}_{\text{exp}}$. Then any quasi-isometry $N \wr G \longrightarrow M \wr H$ is quasi-$\frac{m}{n}$-to-one.

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The class $\mathcal{M}_{\text{exp}}$ has been introduced very recently by Bensaid-Genevois-Tessera, in their study of quasi-isometries of such wreath products. It contains many amenable groups:

- $\mathrm{BS}(1, n)$ ($n \geq 2$), and $\mathrm{BS}(1, n) \times G$ where $G$ is amenable;
- lamplighters over amenable groups, e.g. $F \wr \mathbb{Z}$;
- $\mathrm{SOL}(\mathbb{Z})$, ...

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- $\mathrm{SOL}(\mathbb{Z})$, ...

As a direct consequence of the theorem:

**Corollary [Dum25+]**

We have $\mathrm{Sc}(N \wr G) = \{1\}$ when $N$ has polynomial growth and $G \in \mathcal{M}_{\text{exp}}$ is finitely presented and amenable.

# Algebraic consequences

# Finite-index subgroups

The previous corollary allows to show easily some nice facts. First:

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If $N$ has polynomial growth and $G \in \mathcal{M}_{\exp}$ is finitely presented and amenable, then $N \wr G$ has no proper finite-index subgroups isomorphic to itself.

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Proof: Let $H \leqslant N \wr G$ be such that $H \cong N \wr G$ and $[N \wr G : H] < \infty$. Then, by composing the isomorphism, which is quasi-one-to-one, and the natural inclusion, which is quasi-$\frac{1}{[N \wr G:H]}$-to-one, one gets a quasi-$\frac{1}{[N \wr G:H]}$-to-one quasi-isometry

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Thus $\frac{1}{[N \wr G:H]} \in \mathrm{Sc}(N \wr G) = \{1\}$, so $[N \wr G : H] = 1$ and $H = N \wr G$. □

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Thus $\frac{1}{[N \wr G:H]} \in \mathrm{Sc}(N \wr G) = \{1\}$, so $[N \wr G : H] = 1$ and $H = N \wr G$. $\square$

In fact, more generally:

**Corollary**

If $H_1, H_2 \leqslant N \wr G$ have finite-index and are isomorphic, then $[N \wr G : H_1] = [N \wr G : H_2]$.

# Geometric consequences

# Obstructions for iterated wreath products

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$$\mathbb{Z}_n \wr (N_1 \wr G) \longrightarrow \mathbb{Z}_m \wr (N_2 \wr H)$$

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For instance, if $n \geq 2$, there is no quasi-isometry

$$\mathbb{Z}_2 \wr (\mathbb{Z}^2 \wr \mathrm{BS}(1,n)) \longrightarrow \mathbb{Z}_4 \wr (\mathbb{Z}^3 \wr \mathrm{BS}(1,n))$$

and no other invariant seems to be able to distinguish them, they have:

# Obstructions for iterated wreath products

- Same asymptotic dimension, same number of ends;
- Same type of growth, same isoperimetric profile (Erschler '03 [Ers03]);
- Both are not hyperbolic;
- Both do not have the thick bigon property (Genevois-Tessera '24 [GT24]);
- Both do not have Shalom's property $H_{\mathrm{FD}}$ (Brieussel-Zheng '19 [BZ19]);
- Both have linear divergence;
- ...

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